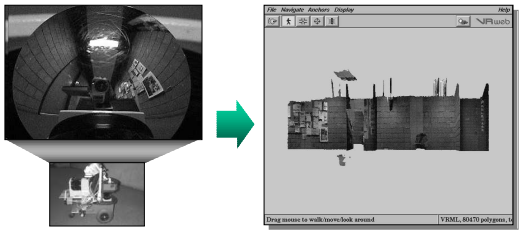
CS329 Probabilistic Robotics

Today's Topic: 3D Mapping with EM

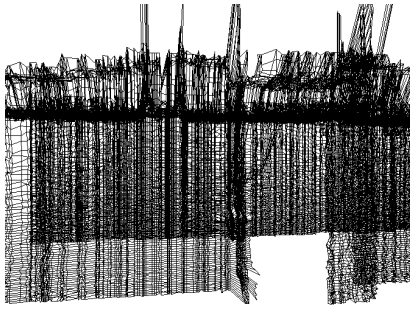
Sebastian Thrun
Carnegie Mellon University
www.cs.cmu.edu/~thrun

The Problem



- Forward-pointed laser (2D information)
- Upward-pointed laser (3D information)
- Panoramic camera

The (Almost) Raw Range Data



3D Mapping

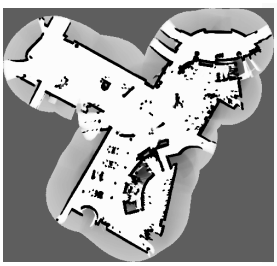
- Observation: World mostly composed of planar surfaces

- Problem: How to generate multi-plane model?

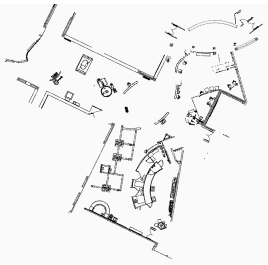
The 3D Multi-Plane Mapping Problem

- Entails a range of problems
  - Parameter estimation: Each surface presently possesses 6 parameters (x, y, z, θ, ϕ, ψ)
  - Outlier identification: Shall we explain every single measurement by a planar surface?
  - Correspondence: What measurement corresponds to what surface?
  - Model selection: How many surfaces should there be?
  - Robot localization: Noise in robot motion has a systematic effect on multiple measurements (this spoils any independence assumption)

Concurrent Mapping and Localization:
To be Discussed in Later Cllases



occupancy grid map



CAD map

The 3D Multi-Plane Mapping Problem

- Entails a range of problems
  - Parameter estimation: Each surface presently possesses 6 parameters $(x, y, z, \theta, \phi, \psi)$
  - Outlier identification: Shall we explain every single measurement by a surface?
  - Correspondence: What measurement corresponds to what surface?
  - Model selection: How many surfaces should there be?
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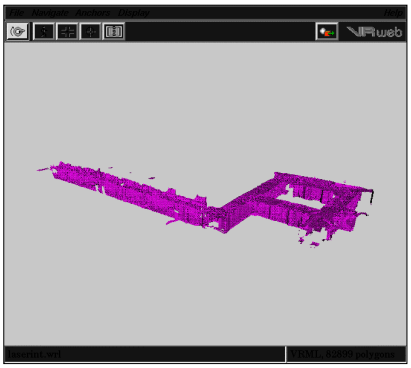
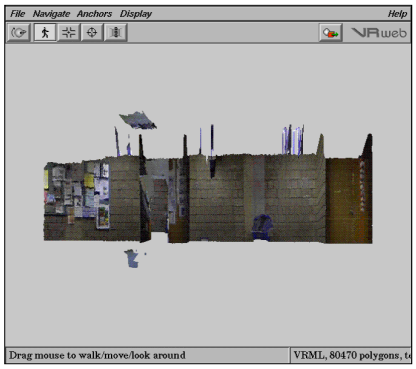
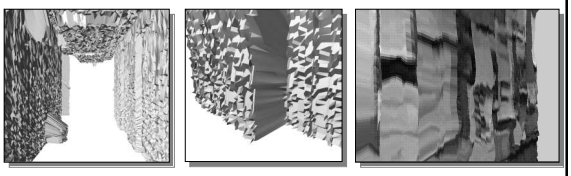
3D Structure Mapping (Raw Data)



3D Texture Mapping (Raw Data)

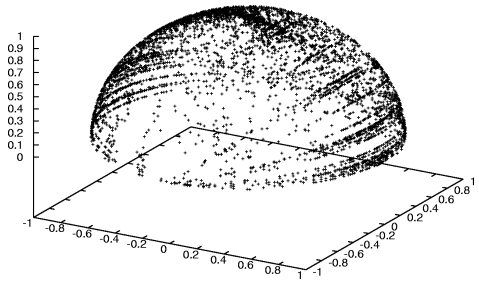


Fine-Grained Structure



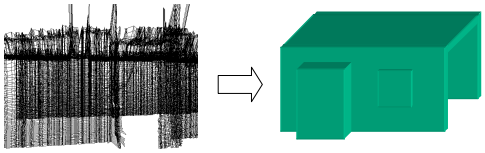
- Can we do better?

Fine-Grained Structure (Wall data): Scatter plot of surface normals



By W. Burgard and D. Hähnel

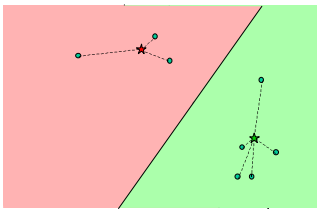
The Remaining Problem



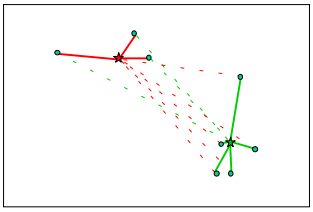
Data

Mostly Multi-Planar model

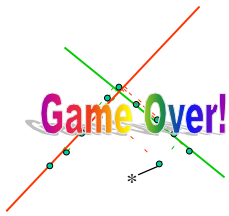
Clustering Data with K-Means



EM: Generalized K-Means



EM for Fitting Multi-Planar Models

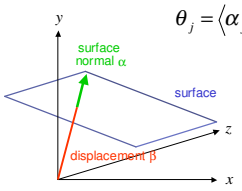


Probabilistic Model

- 3D Model: $\theta = \{\theta_1, \theta_2, \ldots, \theta_J\}$

Planar surface in 3D

$\theta_j = \langle \alpha_j, \beta_j \rangle \in \Re^3 \times \Re$



Distance point-surface

$\text{dist}(\theta_j, z_i) = |\alpha_j \cdot z_i - \beta|$

Measurement Model

- Case 1: Measurement z_i caused by plane θ_j

$$p(z_i \mid \theta_j) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(\alpha_j \cdot z_i - \beta_j)^2}{\sigma^2}}$$

- Case 2: z_i caused by non-planar object

$$p(z_i \mid \theta_*) = \frac{1}{z_{\max}} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\ln\frac{z_{\max}^2}{2\pi\sigma^2}}$$

Measurement Model

$$p(z_i \mid \theta, c_1, \ldots, c_J, c_*) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(c_* \ln\frac{z_{\max}^2}{2\pi\sigma^2} + \sum_{j=1}^{J} c_j \frac{(\alpha_j \cdot z_i - \beta_j)^2}{\sigma^2}\right)}$$

correspondence variables C:

$c_*, c_j \in \{0,1\}$

$c_* + \sum_{j=1}^{J} c_j = 1$

$$\Rightarrow p(Z \mid \theta, C) = \prod \frac{1}{\sqrt{\quad^2}} e^{-\frac{1}{2}\left(c_* \ln\frac{z_{\max}^2}{2\pi\sigma^2} + \sum_{j=1}^{J} c_j \frac{(\alpha_j \cdot z_i - \beta_j)^2}{\sigma^2}\right)}$$

Expected Log-Likelihood Function

$$p(Z\,|\,\theta,C) = \prod_{i=1} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2}\left(c_{i*}\ln\frac{z_{\max}^2}{2\pi\sigma^2}+\sum_{j=1}^{J} c_{ij}\frac{(\alpha_j\cdot z_i-\beta_j)^2}{\sigma^2}\right)}$$

…after some simple math

$$E_c[\ln p(Z,C\,|\,\theta)\,|\,Z,\theta] = \sum_i \begin{pmatrix} \ln\frac{1}{(J+1)\sqrt{2\pi\sigma^2}} \\ -\frac{1}{2}E[c_{i*}\,|\,Z,\theta]\ln\frac{z_{\max}^2}{2\pi\sigma^2} \\ -\frac{1}{2}E[c_{ij}\,|\,Z,\theta]\frac{(\alpha_j\cdot z_i-\beta_j)^2}{\sigma^2} \end{pmatrix}$$

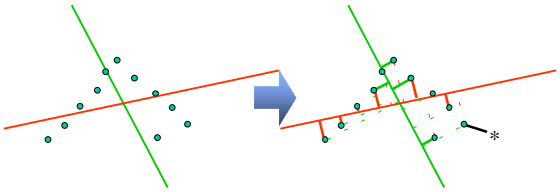
The EM Algorithm

- Define:

$$Q(\theta'\,|\,\theta) = E_c[\ln p(Z,C\,|\,\theta')\,|\,Z,\theta] = \sum_i \begin{pmatrix} \ln\frac{1}{(J+1)\sqrt{2\pi\sigma^2}} \\ -\frac{1}{2}E[c_{i*}\,|\,Z,\theta]\ln\frac{z_{\max}^2}{2\pi\sigma^2} \\ -\frac{1}{2}E[c_{ij}\,|\,Z,\theta]\frac{(\alpha_j'\cdot z_i-\beta_j')^2}{\sigma^2} \end{pmatrix}$$

- Initialization: Pick random θ
- E-Step: Calculate expectations in $Q(\theta^{[n+1]}\,|\,\theta^{[n]})$
- M-Step: Generate New Map $\theta^{[n+1]} = \underset{\theta}{\operatorname{argmax}}\ Q(\theta\,|\,\theta^{[n]})$

The E (Expectation) Step



E-Step

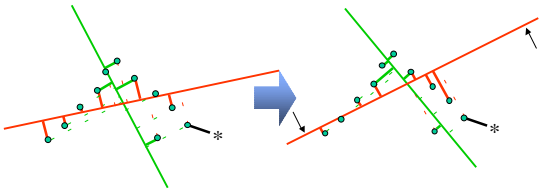
- Calculate expectations with fixed model θ:

$$E[c_{i*}\,|\,Z,\theta^{[n]}] \propto e^{-\frac{1}{2}\ln\frac{z_{\max}^2}{2\pi\sigma^2}}$$

$$E[c_{ij}\,|\,Z,\theta^{[n]}] \propto e^{-\frac{1}{2}\frac{(\alpha_j\cdot z_i-\beta_j)^2}{\sigma^2}}$$

(normalize so that $E[c_{i*}\,|\,Z,\theta^{[n]}]+\sum_{j=1}^{J} E[c_{ij}\,|\,Z,\theta^{[n]}]=1$)

The M (Maximization) Step



M-Step

- Maximize

$$Q(\theta'\,|\,\theta) = E_c[\ln p(Z,C\,|\,\theta')\,|\,Z,\theta] = \sum_i \begin{pmatrix} \ln\frac{1}{(J+1)\sqrt{2\pi\sigma^2}} \\ -\frac{1}{2}E[c_{i*}\,|\,Z,\theta]\ln\frac{z_{\max}^2}{2\pi\sigma^2} \\ -\frac{1}{2}E[c_{ij}\,|\,Z,\theta]\frac{(\alpha_j'\cdot z_i-\beta_j')^2}{\sigma^2} \end{pmatrix}$$

subject to $\alpha_j'\cdot\alpha_j'=1$

- Is equivalent to minimizing

$$\sum_i\sum_j E[c_{ij}\,|\,Z,\theta^{[n]}](\alpha_j'\cdot z_i-\beta_j')^2 \text{ subject to } \alpha_j'\cdot\alpha_j'=1$$

M-Step: Solve via Lagrange Multipliers

Notice: the notation has been simplified

$$\sum_i \sum_{j=1}^{J} E[c_{ij}](\alpha_j \cdot z_i - \beta_j)^2 \to \min \qquad \forall j : \alpha_j \cdot \alpha_j = 1$$

- Define

$$L = \sum_i \sum_{j=1}^{J} E[c_{ij}](\alpha_j \cdot z_i - \beta_j)^2 + \sum_{j=1}^{J} \lambda_j \alpha_j \cdot \alpha_j$$

- And observe that

$$\frac{\partial L}{\partial \alpha_j} = E[c_{ij}](\alpha_j \cdot z_i - \beta_j) z_i - \lambda_j \alpha_j \stackrel{!}{=} 0$$

$$\frac{\partial L}{\partial \beta_j} = E[c_{ij}](\alpha_j \cdot z_i - \beta_j) \stackrel{!}{=} 0$$

$$\alpha_j \cdot \alpha_j = 1$$

M-Step: Solve via Lagrange Multipliers

- Solve for β:

$$\frac{\partial L}{\partial \beta_j} = E[c_{ij}](\alpha_j \cdot z_i - \beta_j) \stackrel{!}{=} 0 \qquad \Rightarrow \beta_j = \frac{\sum_k E[c_{kj}]\alpha_j \cdot z_k}{\sum_k E[c_{kj}]\alpha_j}$$

- Substitute back:

$$\frac{\partial L}{\partial \alpha_j} = E[c_{ij}](\alpha_j \cdot z_i - \beta_j) z_i - \lambda_j \alpha_j \stackrel{!}{=} 0$$

$$\Rightarrow \frac{\partial L}{\partial \alpha_j} = E[c_{ij}](\alpha_j \cdot z_i - \frac{\sum_k E[c_{kj}]\alpha_j \cdot z_k}{\sum_k E[c_{kj}]\alpha_j}) z_i - \lambda_j \alpha_j \stackrel{!}{=} 0$$

- Is of the linear form:

$$A_j \alpha = \lambda_j \alpha_j$$

- Solution: two Eigenvectors of A with smallest Eigenvalues.

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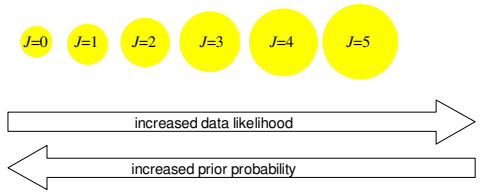
Choosing the "Right" Number of Planes: AIC



$$\log p(J \mid d) = const + \log p(d \mid J) + \log p(J)$$

Model Selection

Approximately every 20 iterations of EM:

- Start new surfaces
  - Near any set of collinear measurements
- Terminate unsupported surfaces
  - If not supported by enough measurements
  - If density of measurements too low
  - If two planes are too close to each other

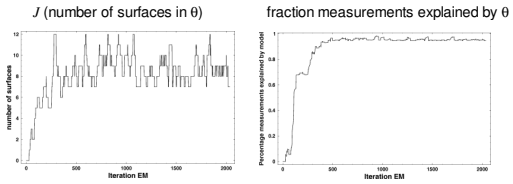
Quality of 3D model

Without EM | With EM

Quality of 3D model

Without EM | With EM

Numerical Results



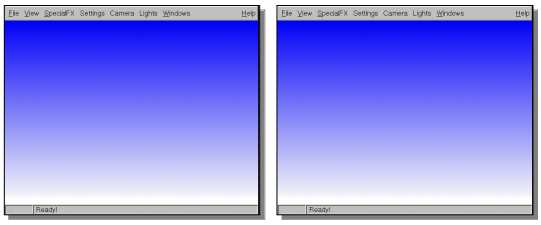
Advantages and Limitations

- Increased compactness, accuracy ☺
- Not real-time ☹
- Only flat surfaces ☹

Towards Online EM

- Each Measurement is only considered in $\leq N$ E-steps
- When doing so, each measurement only compared to $\leq K$ surfaces
- Each surface calculated from $\leq M$ measurements
- New surfaces only started from recent measurements, discarded if not growing fast
- …with small N, K, and M

Online EM and Model Selection



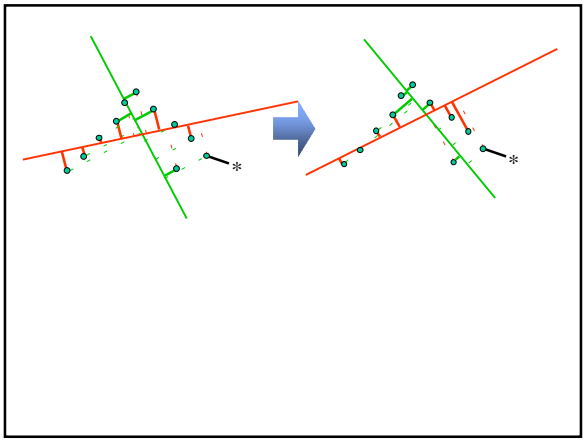
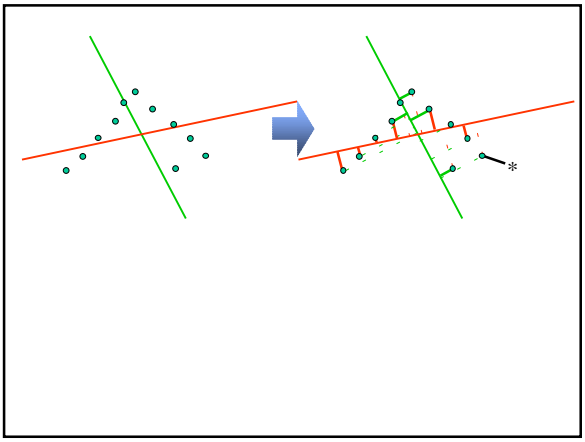
raw data | mostly planar map with Online EM

Relation to Bayes Filters??

- Maximizes posterior likelihood

$$\operatorname*{argmax}_{\theta} E_{c_{1..t}}[\log p(z_{1..t}, c_{1..t} | \theta) | z_{1..t}, \theta] + \log p(\theta)$$
$$\approx \operatorname*{argmax}_{\theta} p(\theta | z_{1..t})$$

instead of computing posterior incrementally

$$p(\theta | z_{1..t}) = f(p(\theta | z_{1..t-1}), z_t)$$

- What does this imply?