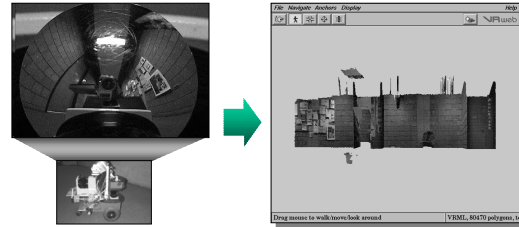


CS329 Probabilistic Robotics

Today's Topic: 3D Mapping with EM

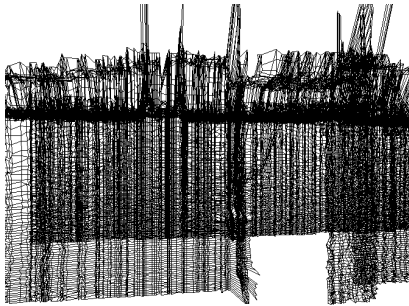
Sebastian Thrun
Carnegie Mellon University
www.cs.cmu.edu/~thrun

The Problem



- Forward-pointed laser (2D information)
- Upward-pointed laser (3D information)
- Panoramic camera

The (Almost) Raw Range Data



3D Mapping

- Observation: World mostly composed of planar surfaces

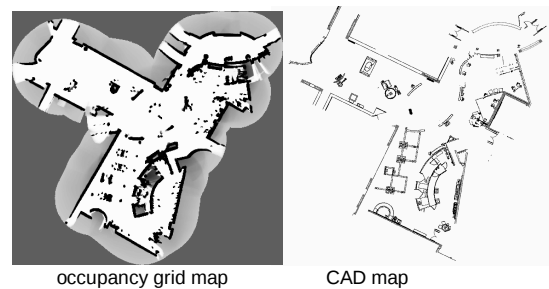


- Problem: How to generate multi-plane model?

The 3D Multi-Plane Mapping Problem

- Entails a range of problems
 - **Parameter estimation:** Each surface presently possesses 6 parameters ($x, y, z, \theta, \phi, \psi$)
 - **Outlier identification:** Shall we explain every single measurement by a planar surface?
 - **Correspondence:** What measurement corresponds to what surface?
 - **Model selection:** How many surfaces should there be?
 - **Robot localization:** Noise in robot motion has a systematic effect on multiple measurements (this spoils any independence assumption)

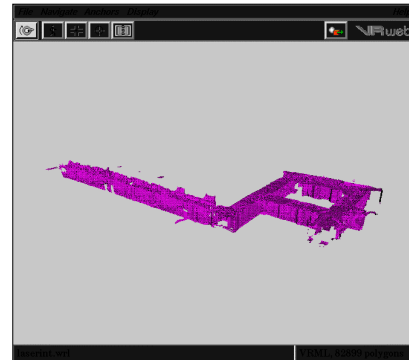
Concurrent Mapping and Localization: To be Discussed in Later Classes



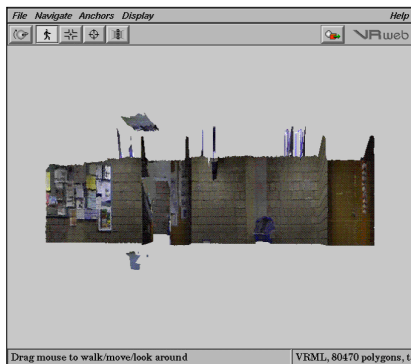
The 3D Multi-Plane Mapping Problem

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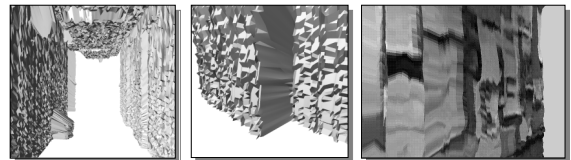
3D Structure Mapping (Raw Data)



3D Texture Mapping (Raw Data)

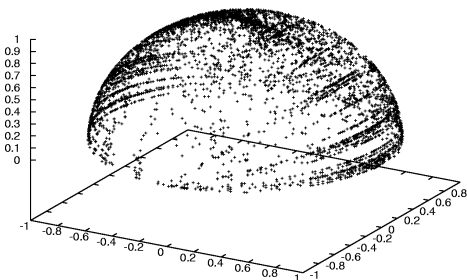


Fine-Grained Structure



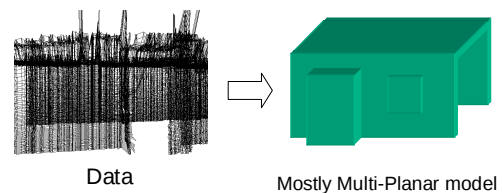
Can we do better?

Fine-Grained Structure (Wall data): Scatter plot of surface normals

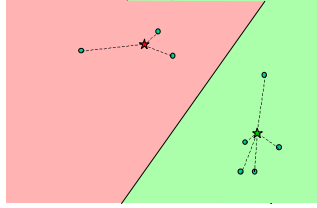


By W. Burgard and D. Hähnel

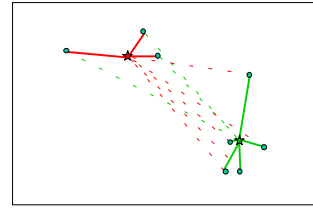
The Remaining Problem



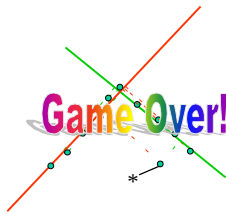
Clustering Data with K-Means



EM: Generalized K-Means



EM for Fitting Multi-Planar Models



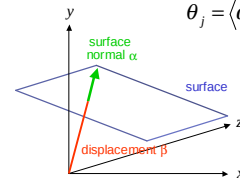
Probabilistic Model

3D Model:

$$\theta = \{\theta_1, \theta_2, \dots, \theta_J\}$$

Planar surface in 3D

$$\theta_j = \langle \alpha_j, \beta_j \rangle \in \mathbb{R}^3 \times \mathbb{R}$$



$$\text{Distance point-surface} \quad \text{dist}(\theta_j, z_i) = |\alpha_j \cdot z_i - \beta|$$

Measurement Model

- Case 1: Measurement z_i caused by plane θ_j

$$p(z_i | \theta_j) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2} \frac{(\alpha_j \cdot z_i - \beta_j)^2}{\sigma^2}}$$

- Case 2: z_i caused by non-planar object

$$p(z_i | \theta_*) = \frac{1}{z_{\max}} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2} \ln \frac{z_{\max}^2}{2\pi\sigma^2}}$$

Measurement Model

$$p(z_i | \theta, c_1, \dots, c_J, c_*) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2} \left(c_* \ln \frac{z_{\max}^2}{2\pi\sigma^2} + \sum_{j=1}^J c_j \frac{(\alpha_j \cdot z_i - \beta_j)^2}{\sigma^2} \right)}$$

correspondence variables C:

$$c_*, c_j \in \{0, 1\}$$

$$c_* + \sum_{j=1}^J c_j = 1$$

$$\Rightarrow p(Z | \theta, C) = \prod \frac{1}{z_{\max}} e^{-\frac{1}{2} \left(c_* \ln \frac{z_{\max}^2}{2\pi\sigma^2} + \sum_{j=1}^J c_j \frac{(\alpha_j \cdot z_i - \beta_j)^2}{\sigma^2} \right)}$$

Expected Log-Likelihood Function

$$p(Z|\theta, C) = \prod_{i=1}^J \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(c_r \ln \frac{z_{\max}^2}{2\pi\sigma^2} + \sum_{j=1}^J c_{ij} \frac{(\alpha_j z_i - \beta_j)^2}{\sigma^2}\right)}$$

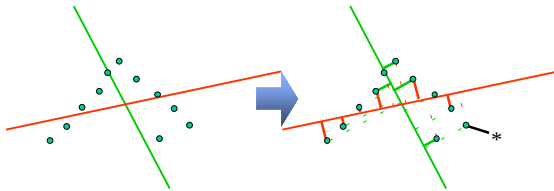
...after some simple math

$$E_c[\ln p(Z, C|\theta)|Z, \theta] = \sum_i \left(\begin{aligned} &\ln \frac{1}{(J+1)\sqrt{2\pi\sigma^2}} \\ & - \frac{1}{2} E[c_r|Z, \theta] \ln \frac{z_{\max}^2}{2\pi\sigma^2} \\ & - \frac{1}{2} E[c_{ij}|Z, \theta] \frac{(\alpha_j z_i - \beta_j)^2}{\sigma^2} \end{aligned} \right)$$

The EM Algorithm

- Define: $Q(\theta|\theta) = E_c[\ln p(Z, C|\theta')|Z, \theta] = \sum_i \left(\begin{aligned} &\ln \frac{1}{(J+1)\sqrt{2\pi\sigma'^2}} \\ & - \frac{1}{2} E[c_r|Z, \theta] \ln \frac{z_{\max}^2}{2\pi\sigma'^2} \\ & - \frac{1}{2} E[c_{ij}|Z, \theta] \frac{(\alpha_j z_i - \beta_j)^2}{\sigma'^2} \end{aligned} \right)$
- Initialization: Pick random θ
- E-Step: Calculate expectations in $Q(\theta^{[n+1]}|\theta^{[n]})$
- M-Step: Generate New Map $\theta^{[n+1]} = \underset{\theta}{\operatorname{argmax}} Q(\theta|\theta^{[n]})$

The E (Expectation) Step



E-Step

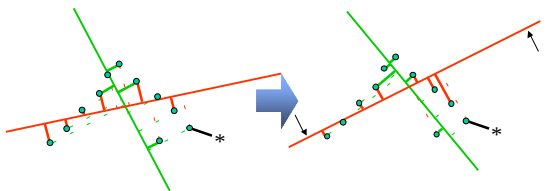
- Calculate expectations with fixed model θ :

$$E[c_r|Z, \theta^{[n]}] \propto e^{-\frac{1}{2} \ln \frac{z_{\max}^2}{2\pi\sigma^2}}$$

$$E[c_{ij}|Z, \theta^{[n]}] \propto e^{-\frac{1}{2} \frac{(\alpha_j z_i - \beta_j)^2}{\sigma^2}}$$

(normalize so that $E[c_r|Z, \theta^{[n]}] + \sum_{j=1}^J E[c_{ij}|Z, \theta^{[n]}] = 1$)

The M (Maximization) Step



M-Step

- Maximize

$$Q(\theta|\theta) = E_c[\ln p(Z, C|\theta')|Z, \theta] = \sum_i \left(\begin{aligned} &\ln \frac{1}{(J+1)\sqrt{2\pi\sigma'^2}} \\ & - \frac{1}{2} E[c_r|Z, \theta] \ln \frac{z_{\max}^2}{2\pi\sigma'^2} \\ & - \frac{1}{2} E[c_{ij}|Z, \theta] \frac{(\alpha_j z_i - \beta_j)^2}{\sigma'^2} \end{aligned} \right)$$

subject to $\alpha_j' \alpha_j' = 1$

Is equivalent to minimizing

$$\sum_{i=1}^J E[c_{ij}|Z, \theta^{[n]}] (\alpha_j' z_i - \beta_j')^2 \text{ subject to } \alpha_j' \alpha_j' = 1$$

M-Step: Solve via Lagrange Multipliers

Notice: the notation has been simplified

$$\sum_i \sum_{j=1}^J E[c_{ij}] (\alpha_j \cdot z_i - \beta_j)^2 \rightarrow \min \quad \forall j: \alpha_j \cdot \alpha_j = 1$$

Define

$$L = \sum_i \sum_{j=1}^J E[c_{ij}] (\alpha_j \cdot z_i - \beta_j)^2 + \sum_{j=1}^J \lambda_j \alpha_j \cdot \alpha_j$$

And observe that

$$\frac{\partial L}{\partial \alpha_j} = E[c_{ij}] (\alpha_j \cdot z_i - \beta_j) z_i - \lambda_j \alpha_j \stackrel{!}{=} 0$$

$$\frac{\partial L}{\partial \beta_j} = E[c_{ij}] (\alpha_j \cdot z_i - \beta_j) \stackrel{!}{=} 0$$

$$\alpha_j \cdot \alpha_j = 1$$

M-Step: Solve via Lagrange Multipliers

Solve for β :

$$\frac{\partial L}{\partial \beta_j} = E[c_{ij}] (\alpha_j \cdot z_i - \beta_j) \stackrel{!}{=} 0 \Rightarrow \beta_j = \frac{\sum_k E[c_{kj}] \alpha_j \cdot z_k}{\sum_k E[c_{kj}] \alpha_j \cdot \alpha_j}$$

Substitute back:

$$\frac{\partial L}{\partial \alpha_j} = E[c_{ij}] (\alpha_j \cdot z_i - \beta_j) z_i - \lambda_j \alpha_j \stackrel{!}{=} 0$$

$$\Rightarrow \frac{\partial L}{\partial \alpha_j} = E[c_{ij}] (\alpha_j \cdot z_i - \frac{\sum_k E[c_{kj}] \alpha_j \cdot z_k}{\sum_k E[c_{kj}] \alpha_j \cdot \alpha_j}) z_i - \lambda_j \alpha_j \stackrel{!}{=} 0$$

Is of the linear form:

$$A_j \alpha_j = \lambda_j \alpha_j$$

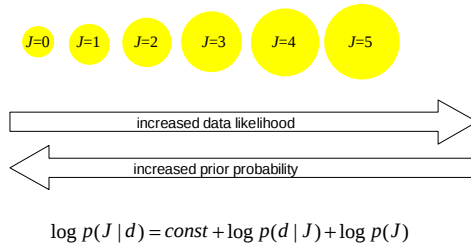
Solution: two Eigenvectors of A with smallest Eigenvalues.

The 3D Multi-Plane Mapping Problem

Entails a range of problems

- Parameter estimation:** Each surface presently possesses parameters $(x, y, z, \theta, \phi, \psi)$
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Choosing the "Right" Number of Planes: AIC

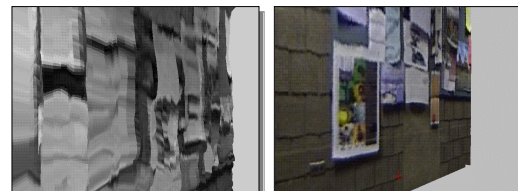


Model Selection

Approximately every 20 iterations of EM:

- Start new surfaces
 - Near any set of collinear measurements
- Terminate unsupported surfaces
 - If not supported by enough measurements
 - If density of measurements too low
 - If two planes are too close to each other

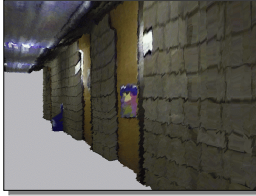
Quality of 3D model



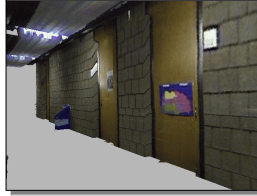
Without EM

With EM

Quality of 3D model

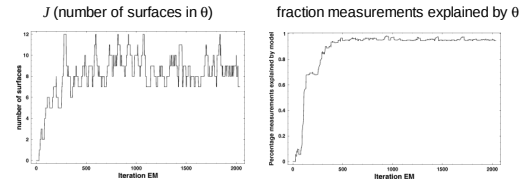


Without EM



With EM

Numerical Results



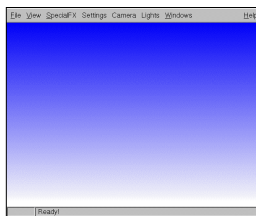
Advantages and Limitations

- Increased compactness, accuracy ☺
- Not real-time ☹
- Only flat surfaces ☹

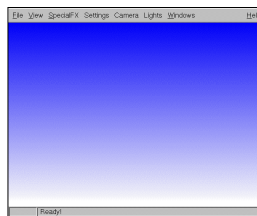
Towards Online EM

- Each Measurement is only considered in $\leq N$ E-steps
- When doing so, each measurement only compared to $\leq K$ surfaces
- Each surface calculated from $\leq M$ measurements
- New surfaces only started from recent measurements, discarded if not growing fast
- ...with small N , K , and M

Online EM and Model Selection



raw data



mostly planar map with Online EM

Relation to Bayes Filters??

- Maximizes posterior likelihood

$$\begin{aligned} & \arg\max_{\theta} E_{q_{\theta}} [\log p(z_{1:t}, c_{1:t} | \theta) | z_{1:t}, \theta] + \log p(\theta) \\ & \approx \arg\max_{\theta} p(\theta | z_{1:t}) \end{aligned}$$

instead of computing posterior incrementally

$$p(\theta | z_{1:t}) = f(p(\theta | z_{1:t-1}), z_t)$$

- What does this imply?

