Computer Science 226

Solution to Assignment #1
(Winter 2006)

January 19, 2006

1. Suppose we live at a place where days are either sunny, cloudy, or rainy. The weather transition function is a Markov chain with the following transition table:

		tomorrow will be…		
		sunny	cloudy	rainy
	sunny	.8	.2	0
today it's…	cloudy	.4	.4	.2
	rainy	.2	.6	.2

   (a) Suppose Day 1 is a sunny day. What is the probability of the following sequence of days: Day2 = *cloudy*, Day3 = *cloudy*, Day4 = *rainy*?

   Answer The probability is $0.2 \times 0.4 \times 0.2 = 0.016$

   Grading ✓ Evaluate an instantiation of a probability distribution.

   (b) Write a simulator that can randomly generate sequences of "weathers" from this state transition function.

   Answer You may use any language you wish to implement the simulator.

```
% MATLAB script
% y = sample(x) returns y sampled from distribution x
X_today = [ 0 1 0 ]';  % Cloudy
T = [ 8 2 0 ; 4 4 2 ; 2 6 2 ]' / 10;
X_tomorrow = sample( T * X_today );
if ( X_tomorrow(1) == 1 )
  disp( 'sunny' );
elseif ( X_tomorrow(2) == 1 )
  disp( 'cloudy' );
elseif ( X_tomorrow(3) == 1 )
  disp( 'rainy' );
end
```

   Grading ✓ Sample from a probability distribution.

   (c) Use your simulator to determine the stationary distribution of this Markov chain. The stationary distribution measures the probability that a random day will be sunny, cloudy, or rainy.

Answer Stationary distribution can be approximated by the frequency of weather far into the future.

```
% MATLAB Script
% Burning-in
% (i.e. Remove samples that may be affected by initial condition)
for ( n = 1 : 10000 )
    X = sample( T * X );
end

% Sampling
X_tally = [ 0 0 0 ]';
for ( n = 1 : 10000 )
    X = sample( T * X );
    X_tally = X_tally + X;
end
X_tally = X_tally / 10000;
```

The above script produced $X_{\text{tally}} = (0.6463, 0.2876, 0.0661)^T$

Grading ✓ Numerically estimate stationary distribution.

(d) Can you devise a closed-form solution to calculating the stationary distribution based on the state transition matrix above?

Answer Let x_t be the weather probability distribution on day t and let T be the weather transition matrix. Then, $x_{t+1} = Tx_t$. When we reach a stationary distribution, $x_{t+1} = x_t = \bar{x}$ which implies that $\bar{x} = T\bar{x}$. In other words, the stationary distribution $\bar{x}$ is the eigenvector with a corresponding eigenvalue of 1. In our case, $\bar{x} = (0.642857, 0.285714, 0.071428)^T$.

Grading ✓ Analytically compute stationary distribution.

(e) What is the entropy of the stationary distribution?

Answer Entropy of the system is $-\sum_i \bar{x}_i \log_2(\bar{x}_i) = 1.198117$

Grading ✓ Definition of entropy.

(f) Using Bayes rule, compute the probability table of yesterday's weather given today's weather. (It is okay to provide the probabilities numerically, and it is also okay to rely on results from previous questions in this exercise.)

Answer By Bayes rule

$$p(x_{t-1}|x_t) = \eta p(x_t|x_{t-1})p(x_{t-1})$$

$$= \frac{p(x_t|x_{t-1})p(x_{t-1})}{\sum_{x'_{t-1}} p(x_t|x'_{t-1})p(x'_{t-1})}$$

Using stationary distribution as prior, we get

		yesterday was...		
		sunny	cloudy	rainy
today it's...	sunny	0.8	0.17778	0.02222
	cloudy	0.45	0.4	0.15
	rainy	0	0.8	0.2

Using uniform distribution as prior, we get

		yesterday was...		
		sunny	cloudy	rainy
	sunny	4/7	2/7	1/7
today it's...	cloudy	1/6	1/3	1/2
	rainy	0	1/2	1/2

Grading ✓ Bayes rule

(g) Suppose we added seasons to our model. The state transition function above would only apply to the Summer, whereas different ones would apply to Winter, Spring, and Fall. Would this violate the Markov property of this process? Explain your answer.

Answer The system still satisfies Markov assumption. It can be viewed as a time-varying Markov Chain (though time-varying, the value of x_{t+1} is still determined by x_t only).

Alternatively, add season s_t as an additional state variable of four values. By extending the weather transition matrix to 12×12, we can still account for all possible transitions from $\{x_t, s_t\}$ to $\{x_{t+1}, s_{t+1}\}$

$$p(x_{t+1}, s_{t+1} | x_{1:t}, s_{1:t}) = p(x_{t+1}, s_{t+1} | x_t, s_t)$$

Grading ✓ Independence and Markov assumption

2. Suppose that we cannot observe the weather directly, but instead rely on a sensor. The problem is that our sensor is noisy. Its measurements are governed by the following measurement model:

		our sensor tells us...		
		sunny	cloudy	rainy
	sunny	.6	.4	0
the actual weather is...	cloudy	.3	.7	0
	rainy	0	0	1

(a) Suppose Day 1 is sunny (this is known for a fact), and in the subsequent four days our sensor observes *cloudy, cloudy, rainy, sunny.* What is the probability that Day 5 is indeed sunny as predicted by our sensor?

Answer By applying Bayes' rule and Markov assumption:

$$\begin{aligned} P(x_5|x_1, z_{2:5}) &= \eta P(z_5|x_5, x_1, z_{2:4}) P(x_5|x_1, z_{2:4}) \\ &= \eta P(z_5|x_5) P(x_5|x_1, z_{2:4}) \end{aligned}$$

Then

$$\begin{aligned} P(x_5|x_1, z_{2:4}) &= \sum_{x_4} P(x_4, x_5|x_1, z_{2:4}) \\ &= \sum_{x_4} P(x_5|x_4, x_1, z_{2:4}) P(x_4|x_1, z_{2:4}) \\ &= \sum_{x_4} P(x_5|x_4) P(x_4|x_1, z_{2:4}) \end{aligned}$$

Note, however, that $z_4 = rainy$ implies that $x_4 = rainy$.

$$
\begin{aligned}
P(x_5|x_1, z_{2:4}) &= \sum_{x_4} P(x_5|x_4)P(x_4|x_1, z_{2:4}) \\
&= P(x_5|x_4 = rainy) \cdot 1
\end{aligned}
$$

Altogether

$$
\begin{aligned}
& P(x_5 = sunny|x_1, z_{2:3}, z_4 = rainy, z_5 = sunny) \\
&= \eta P(z_5 = sunny|x_5 = sunny)P(x_5 = sunny|x_4 = rainy) \\
&= \frac{P(z_5 = sunny|x_5 = sunny)P(x_5 = sunny|x_4 = rainy)}{\sum_{x_5'} P(z_5 = sunny|x_5')P(x_5'|x_4 = rainy)} \\
&= \frac{0.6 \cdot 0.2}{0.6 \cdot 0.2 + 0.3 \cdot 0.6 + 0} \\
&= 0.4
\end{aligned}
$$

Grading ✓ Bayes' rule

✓ Incorporate state transition probability $P(x_{t+1}|x_t)$

✓ Incorporate measurement transition probability $P(z_t|x_t)$.

✓ Correct probability

(b) Once again, suppose Day 1 is known to be sunny. At Days 2 through 4, the sensor measures *sunny, sunny, rainy.* For each of the Days 2 through 4, what is the most likely weather on that day? Answer the question in two ways: one in which only the data available to the day in question is used, and one in hindsight, where data from future days is also available.

Answer Day 2: 88.9% sunny, 11.1% cloudy, 0% rainy.

$$
\begin{aligned}
P(x_2|x_1, z_2) &= \eta P(z_2|x_2, x_1)P(x_2|x_1) \\
&= \eta P(z_2|x_2)P(x_2|x_1) \\
&= \eta \begin{pmatrix} 0.6 \\ 0.3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0.8 \\ 0.2 \\ 0 \end{pmatrix} \\
&= \eta \begin{pmatrix} 0.48 \\ 0.06 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 8/9 \\ 1/9 \\ 0 \end{pmatrix}
\end{aligned}
$$

Day 2 with data from future days: 80% sunny, 20% cloudy, 0% rainy

$$
\begin{aligned}
P(x_2|x_1, z_{2:4}) &= \eta P(x_2|x_1) P(z_{2:4}|x_2, x_1) \\
&= \eta P(x_2|x_1) P(z_{2:4}|x_2) \\
&= \eta P(x_2|x_1) P(z_2|z_{3:4}, x_2) P(z_{3:4}|x_2) \\
&= \eta P(x_2|x_1) P(z_2|x_2) P(z_{3:4}|x_2) \\
&= \eta P(x_2|x_1) P(z_2|x_2) \sum_{x_3} P(x_3, z_{3:4}|x_2) \\
&= \eta P(x_2|x_1) P(z_2|x_2) \sum_{x_3} P(x_3|x_2) P(z_{3:4}|x_3, x_2) \\
&= \eta P(x_2|x_1) P(z_2|x_2) \sum_{x_3} P(x_3|x_2) P(z_{3:4}|x_3) \\
&= \eta P(x_2|x_1) P(z_2|x_2) \sum_{x_3} P(x_3|x_2) P(z_3|x_3) P(z_4|z_3, x_3) \\
&= \eta P(x_2|x_1) P(z_2|x_2) \sum_{x_3} P(x_3|x_2) P(z_3|x_3) P(z_4|x_3) \\
&= \eta P(x_2|x_1) P(z_2|x_2) \sum_{x_3} P(x_3|x_2) P(z_3|x_3) \sum_{x_4} P(x_4, z_4|x_3) \\
&= \eta P(x_2|x_1) P(z_2|x_2) \sum_{x_3} P(x_3|x_2) P(z_3|x_3) \sum_{x_4} P(x_4|x_3) P(z_4|x_4, x_3) \\
&= \eta P(x_2|x_1) P(z_2|x_2) \sum_{x_3} P(x_3|x_2) P(z_3|x_3) \sum_{x_4} P(x_4|x_3) P(z_4|x_4) \\
&= \eta P(x_2|x_1) P(z_2|x_2) \sum_{x_3} P(x_3|x_2) P(z_3|x_3) \sum_{x_4} P(x_4|x_3) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}_{P(x_4|z_4)} \\
&= \eta P(x_2|x_1) P(z_2|x_2) \sum_{x_3} P(x_3|x_2) P(z_3|x_3) \cdot \begin{pmatrix} 0 \\ 0.2 \\ 0.2 \end{pmatrix}_{P'(x_3|z_4)} \\
&= \eta P(x_2|x_1) P(z_2|x_2) \sum_{x_3} P(x_3|x_2) \cdot \begin{pmatrix} 0 \\ 0.06 \\ 0 \end{pmatrix}_{P'(x_3|z_3, z_4)} \\
&= \eta P(x_2|x_1) P(z_2|x_2) \cdot \begin{pmatrix} 0.012 \\ 0.024 \\ 0.036 \end{pmatrix}_{P'(x_2|z_3, z_4)} \\
&= \eta \begin{pmatrix} 0.8 \\ 0.2 \\ 0 \end{pmatrix}_{P'(x_2|x_1)} \cdot \begin{pmatrix} 0.6 \\ 0.3 \\ 0 \end{pmatrix}_{P'(x_2|z_2)} \cdot \begin{pmatrix} 0.012 \\ 0.024 \\ 0.036 \end{pmatrix}_{P'(x_2|z_3, z_4)} \\
&= \eta \begin{pmatrix} 0.00576 \\ 0.00144 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0.8 \\ 0.2 \\ 0 \end{pmatrix}
\end{aligned}
$$

I misinterpreted "future days" as "the following day" when posting the FAQ on the web. In which case, probability of weather on day 2 given data from only day 3 is: 92.3% sunny, 7.7% cloudy, 0.0% rainy.

$$
\begin{aligned}
P(x_2|x_1, z_2, z_3) &= \eta P(x_2|x_1)P(z_2, z_3|x_2, x_1) \\
&= \eta P(x_2|x_1)P(z_2, z_3|x_2) \\
&= \eta P(x_2|x_1)P(z_2|z_3, x_2)P(z_3|x_2) \\
&= \eta P(x_2|x_1)P(z_2|x_2)P(z_3|x_2) \\
&= \eta P(x_2|x_1)P(z_2|x_2)\sum_{x_3} P(x_3, z_3|x_2) \\
&= \eta P(x_2|x_1)P(z_2|x_2)\sum_{x_3} P(x_3|x_2)P(z_3|x_3, x_2) \\
&= \eta P(x_2|x_1)P(z_2|x_2)\sum_{x_3} P(x_3|x_2)P(z_3|x_3) \\
&= \eta P(x_2|x_1)P(z_2|x_2)\sum_{x_3} P(x_3|x_2)\begin{pmatrix} 0.6 \\ 0.3 \\ 0 \end{pmatrix}_{P'(x_3|z_3)} \\
&= \eta P(x_2|x_1)P(z_2|x_2)\begin{pmatrix} 0.60 \\ 0.36 \\ 0 \end{pmatrix}_{P'(x_2|z_3)} \\
&= \eta \begin{pmatrix} 0.8 \\ 0.2 \\ 0 \end{pmatrix}_{P'(x_2|x_1)} \cdot \begin{pmatrix} 0.6 \\ 0.3 \\ 0 \end{pmatrix}_{P'(x_2|z_2)} \cdot \begin{pmatrix} 0.54 \\ 0.36 \\ 0 \end{pmatrix}_{P'(x_2|z_3)} \\
&= \eta \begin{pmatrix} 2592/2808 \\ 216/2808 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0.9231 \\ 0.0769 \\ 0 \end{pmatrix}
\end{aligned}
$$

Day 3: 87.2% sunny, 12.8% cloudy, 0% rainy.

$$\begin{aligned}
P(x_3|x_1, z_{2:3}) &= \eta P(z_3|x_3, x_1, z_2) P(x_3|x_1, z_2) \\
&= \eta P(z_3|x_3) \sum_{x_2} P(x_3, x_2|x_1, z_2) \\
&= \eta P(z_3|x_3) \sum_{x_2} P(x_3|x_2) P(x_2|x_1, z_2) \\
&= \eta P(z_3|x_3) \sum_{x_2} P(x_3|x_2) \begin{pmatrix} 8 \\ 1 \\ 0 \end{pmatrix}_{P'(x_2|x_1,z_2)} \\
&= \eta P(z_3|x_3) \begin{pmatrix} 6.8 \\ 2.0 \\ 0.2 \end{pmatrix}_{P'(x_3|x_1,z_2)} \\
&= \eta \begin{pmatrix} 0.6 \\ 0.3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 6.8 \\ 2.0 \\ 0.2 \end{pmatrix} \\
&= \begin{pmatrix} 408/468 \\ 60/468 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.8718 \\ 0.1282 \\ 0 \end{pmatrix}
\end{aligned}$$

Day 3 with data from future days: 0% sunny, 100% cloudy, 0% rainy

$$\begin{aligned}
P(x_3|x_1, z_{2:4}) &= \eta P(x_3|x_1, z_{2:3}) P(z_4|x_3, x_1, z_{2:3}) \\
&= \eta P(x_3|x_2, z_3) P(z_4|x_3) \\
&= \eta \begin{pmatrix} 0.8395 \\ 0.1605 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0.2 \\ 0.2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
\end{aligned}$$

Day 4: 0% sunny, 0% cloudy, 100% rainy

$$\begin{aligned}
P(x_4|x_1, z_{2:4}) &= P(x_4|x_3, z_4) \\
&= \eta P(z_4|x_4) P(x_4|x_3) \\
&= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}$$

Grading ✓ Incorporate initial condition $P(x_k|x_1, \cdots)$

✓ Incorporate previous sensor readings $P(x_k|z_{1:k}, \cdots)$

✓✓✓ Incorporate future sensor readings $P(x_k|z_{k+1:4}, \cdots)$

✓ Correct weather predictions

✓ (bonus) e.g. implementing Bayes filter, etc.

(c) Consider the same situation (Day 1 is sunny, the measurements for Days 2, 3, and 4 are *sunny, sunny, rainy*). What is the most likely sequence of weather for Days 2 through 4? What is the probability of this most likely sequence?

Answer The probability of the sequence of weather is given by

$$P(x_{2:4}|x_1, z_{2:4}) = \eta P(z_{2:4}|x_1, x_{2:4})P(x_{2:4}|x_1)$$

where

$$P(x_{2:4}|x_1) = P(x_4|x_3)P(x_3|x_2)P(x_2|x_1)$$

and

$$P(z_{2:4}|x_1, x_{2:4}) = P(z_4|x_4)P(z_3|x_3)P(z_2|x_2)$$

Hence, the most likely sequence of weather is $sunny, cloudy, rainy$ which has $0.00576 \div (0.00576+0.00144) = 80\%$ of occurring. There is a 20% probability of $cloudy, cloudy, rainy$ and 0% probability for all other sequences of weather.

Grading ✓ Factorize $P(x_{2:4}|x_1, z_{2:4})$

✓ Correct weather sequence and probability reading $P(z_t|x_t)$